# K-semistability of log Fano cone singularities (w/ Yuchen Liu)

$/\mathbb{C}$..

Motivation.

- (Global) $(V, \Delta)$ log Fano, $-(K_V+\Delta)$ ample

    1) (Yau-Tian-Donaldson)

    $\exists$ KE metric on $(V, \Delta)$ $\iff$ K-stability.

    [Chen-Donaldson-Sun, Tian
    Berman-Boucksom-Jonsson, Li, Liu-Xu-Zhuang]

    2) (K-moduli) $\exists$ proj. moduli space param. K-polystable log Fanos.

- (Global-to-local) $L = -r(K_V+\Delta)$ $x \in X = C_a(V, L) = \operatorname{Spec} R(V, L) \circlearrowleft \mathbb{G}_m$, $x$ cone pt

    $x \in X$ is a klt sing. $(X, x)$ is a log Fano cone sing

- (Local) $(X = \operatorname{Spec} R, \xi)$ LFC

    1) (YTD) $\exists$ Calabi-Yau cone metric $\iff$ local K-stability.
    [Collins-Székelyhidi, Li]

    2) (local moduli) moduli of LFCs.

    Boundedness: Xu-Zhuang.

    Properness: Odaka.

- (Riem. geom) $X$ sm. away $x \in X$. $r$ = ref cone metric

    Link $M = \{r=1\}$ Sasaki mfld.

    KE metric on Fanos $\leftrightarrow$ SE metric on $M$ $\leftrightarrow$ CY cone metric on $X$.

Ex $V = \mathbb{P}^1$ $M = S^3$ $X = \mathbb{C}^2$

## Log Fano cone singularities

$X = \operatorname{Spec} R$, $x \in X$ klt, $T \curvearrowright X$, fixing $x \in X$.

$\rightsquigarrow$ $R = \bigoplus_{\alpha \in \Lambda} R_\alpha$, $\Lambda \subseteq M = \operatorname{Hom}(T, \mathbb{G}_m)$ $N := M^\vee$.

Reebcone: $t_{\mathbb{R}}^+ := \{\xi \in N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \xi, \alpha \rangle > 0, \ \forall \alpha \in \Lambda \setminus \{0\}\}$

$\xi \in t_{\mathbb{R}}^+$ is a Reeb field

$(X, T, \xi)$ is a LFC

Ex • $X = C_a(V, L) \circlearrowleft T = \mathbb{G}_m$ $\xi = 1 \in \mathbb{R}_{>0}$ $(X, \mathbb{G}_m, 1)$ is a LFC

• $X = X_\sigma$ affine toric, $\sigma \subseteq \mathbb{R}^n$ $t_{\mathbb{R}}^+ = \operatorname{int}(\sigma)$

e.g. $X = \mathbb{A}^2$ $T = \mathbb{G}_m^2$, $\xi = (1,1), (1, \sqrt{2})$

(local-to-global) If $\xi \in t_{\mathbb{Q}}^+$, then $\langle \xi \rangle = \mathbb{G}_m$ &

$(X \setminus \{x\}) / \langle \xi \rangle = (V, \Delta)$ log Fano pair.

NA perspective:

Each $\xi \in t_{\mathbb{R}}^+$ gives a valuation $v : \mathbb{C}(X) \to \mathbb{R} \cup \{\infty\}$

$v_\xi(f_\alpha \in R_\alpha) := \langle \xi, \alpha \rangle$

$v_\xi(\sum f_\alpha) = \min \{\langle \xi, \alpha \rangle\}$

$v(fg) = v(f) + v(g)$

$v(f+g) \geq \min\{v(f), v(g)\}$

$v|_{\mathbb{C}} = v_{triv}$, i.e. $v(0) = \infty$, $v(a \neq 0) = 1$

Ex $X = C_a(V, L)$ $\quad v \cong E \subseteq \tilde{X} \xrightarrow{Bl_x} X$ $\quad v_\xi = \operatorname{ord}_E$

Fact (Li-Xu). $v_\xi$ is divisorial iff $\xi \in t_{\mathbb{Q}}^+$.

• More generally, $\forall \xi \in t_{\mathbb{R}}^+$, $v_\xi$ is quasi-monomial

Moreover, $\xi \rightsquigarrow$ NA ref metric $r^{NA} = e^{\varphi_\xi}$, where

$\varphi_\xi : X^{an} \to \mathbb{R}$, $\quad \varphi_\xi = \max\left\{ \frac{\log|f_\alpha|}{\langle \xi, \alpha \rangle} \right\}$, $\log|f_\alpha|(v) := -v(f_\alpha)$

<u>NA link</u> $\quad X_0 := \{ r^{NA} = 1 \} = \{ \varphi_\xi = 0 \} = \{ v \in X^{an} \mid v(\mathfrak{m}_x) = 0 \} \overset{cpt}{\subseteq} X^{an}$

<u>Local K-stability</u>

<u>Test configurations</u>

$T \times \mathbb{G}_m \curvearrowright \mathcal{X} = \operatorname{Spec} R$ normal
$\quad\; \| \qquad\qquad\quad \downarrow$
$\langle \eta \rangle \qquad\quad \mathbb{A}^1 \qquad$ s.t.

$\mathcal{X} \times_{\mathbb{A}^1} (\mathbb{A}^1 \setminus \{0\}) \cong X \times (\mathbb{A}^1 \setminus \{0\})$.

[Collins-Székelyhidi]

$\mathrm{Fut}(\mathcal{X}, \xi, \eta) = \mathrm{Fut}(\mathcal{X}_0, \xi_0, \eta_0)$

K-semistability
$\overset{def}{\iff} \mathrm{Fut}(\mathcal{X}, \xi, \eta) \geq 0 \ \forall$ normal TCs.

<u>Special TCs</u>: $\mathcal{X}_0$ is klt

<u>Rmks</u>
- For $\xi \in \mathfrak{t}^+_{\mathbb{Q}}$, local K-stab. = global K-stab
- $\xi \mapsto \mathrm{Fut}(\mathcal{X}, \xi, \eta)$ cts
- No alg. / intersection-theoretic formula for Fut in general

<u>Ex</u>. <u>Thm</u> (Li-Xu '14) To test global K-stability, it suffices to test special TCs.

<u>Q</u> True for local K-stab?

<u>Thm</u> (Liu-W.) True for K-semistability.

<u>NA char. of local K-stab.</u>

$X^{an}$ = Berkovich analytification of $X$ wrt $(\mathbb{C}, v_{triv})$
$\underset{=}{\text{set}}$ {(semi) valuations on $X$ ext. $v_{triv}$ on $\mathbb{C}$}.

{Witt-Nyström,
Boucksom-Hisamoto-Jonsson}:

TCs $\rightsquigarrow$ Filtrations on $R$ $\hookrightarrow$ NA norms on $R$ $\rightsquigarrow$ FS fcn.

$\mathcal{F}$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\chi$ $\quad\quad\quad\quad\quad$ $\varphi$

$\varphi = \max\left\{ \frac{\log|f_\alpha| + \chi(f_\alpha)}{\langle \xi, \alpha \rangle} \right\}$, $X^{an} \to \mathbb{R}$ cts, psh (Chambert-Loir Ducros)

Ex trivial TC $\rightsquigarrow \chi = 0 \rightsquigarrow \varphi_\xi$

$\{\text{TCs}\} \overset{1-1}{\hookrightarrow} \mathcal{H}^{NA} = \{(\text{sm})\text{ psh fcns/potential on } X^{an}\}$.

intersection #s $\leftrightarrow$ NA Monge-Ampère meas. assoc. to $\varphi$

$$MA(\varphi;\xi) = \frac{1}{\mathrm{vol}(\xi)} (d'd''\varphi)^{n-1} \wedge d'd'' \max\{\varphi_\xi, 0\}.$$

Thm (-) • $MA : \mathcal{H}^{NA} \to \mathcal{M}(X_0) = \{\text{prob. meas. on } X^{an} \text{ supp. on } X_0\}$

$$\varphi_{(\mathcal{X},\xi,\eta)} \longmapsto \sum_{\text{finite}} c_v \delta_v, \quad v \in X_0^{T, qm}$$

- $\xi \mapsto MA(\varphi_{(\mathcal{X},\xi,\eta)}; \xi)$ cts
- If $\xi \in \mathfrak{t}^+_{\mathbb{Q}}$ then $i, V^{an} \hookrightarrow X_0 \subset X^{an}$ s.t.

$$i_*(\underset{\text{BHJ}}{\underset{\downarrow}{MA(\phi)}}) = MA(\varphi;\xi)$$

Cor Can define NA fcnls. $D^{NA} : \mathcal{H}^{NA} \to \mathbb{R}$ s.t.

- $D^{NA} \le \mathrm{Fut}$, "=" on sp. TCs.
- $\xi \mapsto D^{NA}(\mathcal{X};\xi,\eta)$ cts

Thm (Liu-W) $(X, T, \xi)$ LFc. TFAE.

1) K-ss.
2) $D^{NA} \ge 0 \;\forall\, \varphi \in \mathcal{H}^{NA}$

3) $\delta \geq 1$ (valuative)

Huang $\Uparrow$

4) K-ss. wrt sp. Tcs

Pf idea

- $MA: \mathcal{E}^{1,NA} (\supseteq \mathcal{H}^{NA}) \to \mathcal{M}(X_0)$
- Translate valuative invariants by "solving MA eqn".